Theory

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Abstract

The simplest possible noncommutative harmonic oscillator in two dimensions is used to quantize the free closed bosonic string in two flat dimensions. The nonsolitonic part of the partition function is not deformed by the introduction of noncommutativity, while the four point function is, preserving, nevertheless, the sl(2,C) invariance. Finally the first Ward identity of the deformed theory is derived.

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In this letter we explore a possible way of quantizing a string based on a noncommutative harmonic oscillator as opposed to the ordinary commutative one [1]. This noncommutativity is neither the one in the D brane worldvolume that arises in the quantization of open strings ending on D branes with background B field [2], nor the one introduced in M-theory compactified on T^2 [3].

To study the modifications that occur in bosonic string theory, the simplest possible noncommutative harmonic oscillator in two dimensions is employed. We assume that the position and momentum operators satisfy the following commutation relations [4]:

$$[q_n^i, p_n^j] = i\hbar\delta_{ij}; \quad [q_n^1, q_n^2] = i\theta/n; \quad [p_n^1, p_n^2] = -in\theta. \tag{1}$$

An operator representation of the $q_n^i(p_n^i)$ that realizes the commutation relations (1) is:

$$\begin{aligned}
q_n^1 &= \sqrt{\frac{\alpha}{2n}}(a_n^1 + {a_n^1}^\dagger) &(2)\\
p_n^1 &= -i\sqrt{\frac{n}{2\alpha}}[(\hbar a_n^1 - i\theta a_n^2) - (\hbar {a_n^1}^\dagger + i\theta {a_n^2}^\dagger)] &(3)\\
q_n^2 &= \sqrt{\frac{1}{2n\alpha}}[(\hbar a_n^2 - i\theta a_n^1) + (\hbar {a_n^2}^\dagger + i\theta {a_n^1}^\dagger)] \equiv \sqrt{\frac{\alpha}{2n}}(A_n + A_n^\dagger) &(4)\\
p_n^2 &= -i\sqrt{\frac{n\alpha}{2}}(a_n^2 - {a_n^2}^\dagger), &(5)
\end{aligned}$$

where $\alpha = \sqrt{\hbar^2 + \theta^2}$. The nontrivial commutation relation satisfied by the creation and annihilation operators in this model is: $[a_n^i, {a_n^j}^\dagger] = \delta_{ij}$, from which, one deduces that $[A_n, A_n^\dagger] = 1$. The Hamiltonian of this harmonic oscillator is $H = \alpha\sum_{i=1}^2 n{a_n^i}^\dagger a_n^i + \text{const.}$ and the time evolution of the modes $a_n^i, {a_n^i}^\dagger$ is found to be $a_n^i(t) = a_n^i e^{-i\frac{\alpha}{\hbar}nt}$ and ${a_n^i}^\dagger(t) = {a_n^i}^\dagger e^{i\frac{\alpha}{\hbar}nt}$.

Consider now the free closed bosonic string in two dimensions. The oscillatory part of the coordinates that satisfies both the equations of motion and the boundary conditions admit a real expansion of the form:

$$\tilde{X}^1(\sigma, t) = \sum_{n>0} q_n^1(t)\cos n\sigma + \bar{q}_n^1(t)\sin n\sigma \tag{6}$$

$$\tilde{X}^2(\sigma, t) = \sum_{n>0} q_n^2(t)\cos n\sigma + \bar{q}_n^2(t)\sin n\sigma. \tag{7}$$

We treat q_n^1, q_n^2 and $\bar{q}_n^1, \bar{q}_n^2$ as two separate systems of two dimensional harmonic oscillators, one based on operators $a_n, a_n^\dagger$ and the other on $b_n, b_n^\dagger$ obeying identical commutation relations. Rescaling the modes by $a_n^i \to a_n^i/\sqrt{n}$, $b_n^i \to b_n^i/\sqrt{n}$ and defining $a_{-n}^1 = -{a_n^1}^\dagger$, $a_{-n}^2 = {a_n^2}^\dagger$, $b_{-n}^1 = {b_n^1}^\dagger$, $b_{-n}^2 = -{b_n^2}^\dagger$ we have:

$$\tilde{X}^1(\sigma, t) = \frac{i}{2}\sum_{n>0}\frac{1}{n}\left(C_n e^{-in(\frac{\alpha}{\hbar}t-\sigma)} + \bar{C}_n e^{-in(\frac{\alpha}{\hbar}t+\sigma)}\right) \tag{8}$$

$$\tilde{X}^2(\sigma, t) = \frac{i}{2}\sum_{n>0}\frac{1}{n}\left(D_n e^{-in(\frac{\alpha}{\hbar}t-\sigma)} + \bar{D}_n e^{-in(\frac{\alpha}{\hbar}t+\sigma)}\right) \tag{9}$$

where

$$C_n = -\sqrt{\frac{\alpha}{2}}\left(b_n^1 + ia_n^1\right) \qquad \bar{C}_n = \sqrt{\frac{\alpha}{2}}\left(b_n^1 - ia_n^1\right) \tag{10}$$

$$D_n = -\sqrt{\frac{\alpha}{2}}\,\mathrm{sign}(n)\,(B_n + iA_n) \qquad \bar{D}_n = \sqrt{\frac{\alpha}{2}}\,\mathrm{sign}(n)\,(B_n - iA_n) \tag{11}$$

$$A_n = \frac{\hbar a_n^2 - i\theta a_n^1}{\alpha} \qquad B_n = \frac{\hbar b_n^2 - i\theta b_n^1}{\alpha}. \tag{12}$$

The C and D modes obey the following commutation relations:

$$[C_n, C_{-n}] = n\alpha; \quad [C_n, D_{-n}] = in\theta\,\mathrm{sign}(n); \quad [D_n, D_{-n}] = n\alpha \tag{13}$$

where the corresponding barred operators satisfy the same commutation relations and commute with the unbarred operators. Performing the Wick rotation and passing to the complex plane ($\tau = -i\frac{\alpha}{\hbar}t$, $w = \tau + i\sigma$, $z = e^{-w}$), we obtain, after including the zero modes,

$$X^1(z) = x_L^1 - ip_L^1 \ln z + i\sum_{n\neq 0}\frac{1}{n}C_n z^{-n} \tag{14}$$

$$\bar{X}^1(\bar{z}) = x_R^1 - ip_R^1 \ln \bar{z} + i\sum_{n\neq 0}\frac{1}{n}\bar{C}_n \bar{z}^{-n} \tag{15}$$

$$X^2(z) = x_L^2 - ip_L^2 \ln z + i\sum_{n\neq 0}\frac{1}{n}D_n z^{-n} \tag{16}$$

$$\bar{X}^2(\bar{z}) = x_R^2 - ip_R^2 \ln \bar{z} + i\sum_{n\neq 0}\frac{1}{n}\bar{D}_n \bar{z}^{-n} \tag{17}$$

where $X^i(\sigma,\tau) = (X^i(z) + \bar{X}^i(\bar{z})/2$.

The regularized Hamiltonian, which is the sum of the Hamiltonians of the individual harmonic oscillators without the constant term, is:

$$H = \sum_{n>0}\left[\frac{\alpha^2}{\hbar^2}(C_{-n}C_n + \bar{C}_{-n}\bar{C}_n + D_{-n}D_n + \bar{D}_{-n}\bar{D}_n) + i\frac{\alpha\theta}{\hbar^2}(D_{-n}C_n - C_{-n}D_n + \bar{D}_{-n}\bar{C}_n - \bar{C}_{-n}\bar{D}_n)\right]. \tag{18}$$

This is the generator of time translations on the oscillatory part or the coordinates X^i. The generator of spatial translations is:

$$P = \sum_{n>0}\left[\frac{\alpha}{\hbar}(C_{-n}C_n - \bar{C}_{-n}\bar{C}_n + D_{-n}D_n - \bar{D}_{-n}\bar{D}_n) + i\frac{\theta}{\hbar}(D_{-n}C_n - C_{-n}D_n - \bar{D}_{-n}\bar{C}_n + \bar{C}_{-n}\bar{D}_n)\right]. \tag{19}$$

Note that the momentum and the Hamiltonian operator commute, as expected.

The nonsolitonic part of the partition function for this prototype theory on the torus is given by:

$$Z = Tr(e^{2\pi i\frac{\tau_1}{\hbar}P}e^{-2\pi\frac{\hbar}{\alpha}\frac{\tau_2}{\hbar}H}) \tag{20}$$

where the trace is taken over the module generated by the negative modes of C and D operators subjected to the commutation relations (including the mixed ones). After evaluating this trace and including the necessary transformation factor $q^{-1/12}$, we get the usual expression as if noncommutativity was absent:

$$Z = \frac{1}{|\eta(\tau)|^4}. \tag{21}$$

To make further progress we need to determine the propagators. These turn out to be:

$$< X^1(z)X^1(w) >=< (x_L^1)^2 > -\alpha \ln(z-w) \tag{22}$$
$$< X^2(z)X^2(w) >=< (x_L^2)^2 > -\alpha \ln(z-w) \tag{23}$$
$$< X^1(z)X^2(w) >=< x_L^1 x_L^2 > -i\theta \ln(z-w) \tag{24}$$
$$< X^2(z)X^1(w) >=< x_L^2 x_L^1 > +i\theta \ln(z-w) \tag{25}$$

In determining (25) we have imposed the following commutation relations among the zero modes:

$$[x_L^1, p_L^1] = i\alpha; \quad [x_L^2, p_L^2] = i\alpha; \quad [x_L^1, p_L^2] = \theta; \quad [x_L^2, p_L^1] = -\theta, \tag{26}$$

in such a way as to avoid unwanted singularities at the origin. Identical relations hold in the antiholomorphic sector.

Regarding the stress-energy tensor, there is no longer a unique generator of conformal transformations for both X^i components. Rather there is the usual tensor $T_1(z) = -\frac{1}{2\alpha} : (\partial_z X^1(z))^2 :$ that generates conformal transformations on $X^1(z)$ but not on $X^2(z)$ and conversely for $T_2(z)$. The algebra of the moments of each stress-energy tensor is the Virasoro algebra with central charge one. The primary fields are the usual ones for each string component.

Our next task is to compute the correlation functions on the sphere. The only interesting two point function is the mixed one:

$$< 0| : e^{ik_1 X^1(z)} :: e^{ik_2 X^2(w)} : |0 >=< 0| : e^{ik_1 x_L^1} :: e^{ik_2 x_L^2} : |0 > (z-w)^{i\theta k_1 k_2} = 0. \tag{27}$$

It vanishes because of the expectation value of the zero modes, unless both k_1, k_2 are 0. The zero mode expectation value also indicates that in higher correlation functions charge conservation must be maintained for each primary field separately. This implies that the first correlation function that will differ from the commutative case is the four point function. It takes the form:

$$<: e^{ikX^1(u)} :: e^{-ikX^1(v)} :: e^{i\lambda X^2(w)} :: e^{-i\lambda X^2(z)} :>= (u-v)^{-\alpha k^2} (w-z)^{-\alpha\lambda^2} \left[\frac{(u-w)(v-z)}{(u-z)(v-w)}\right]^{i\theta k\lambda} . \tag{28}$$

This four point function is invariant under global sl(2,C) transformations, a fact that indicates the dependence of the correlation functions on the complex structure only. Thus the noncommutative quantization procedure we apply gives a nontrivial deformation of the original theory, since the four point function is deformed, and the nice geometric properties of the correlation functions are preserved. However there is a controllable nonlocality that has been introduced by the parameter θ, as can be seen by the position dependence of the commutator of the string components, $[X^1(z), X^2(w)] = -2i\theta \ln(z-w) + i\theta \ln(-1)$.

We complete our discussion by writing down the first Ward identity. It turns out that it can be expressed through the formula:

$$\sum_{i=1}^{N} <: e^{ik_1 X^{n_1}(w_1)} : \cdots \frac{1}{2\pi i} \oint_{w_i} \epsilon(z) T_{n_i}(z) : e^{ik_i X^{n_i}(w_i)} : \cdots : e^{ik_N X^{n_N}(w_N)} :>=$$
$$= \sum_{i=1}^{N} \frac{1}{2\pi i} \oint_{w_i} \left(\frac{\alpha k_i^2}{2} \frac{1}{(z-w_i)^2} + \frac{1}{z-w_i} \partial_{w_i} \right) \epsilon(z) <: e^{ik_1 X^{n_1}(w_1)} : \cdots : e^{ik_N X^{n_N}(w_N)} :> \tag{29}$$

or in the unintegrated form as:

$$
\begin{aligned}
& < T_1(z) : e^{ik_1 X^{n_1}(w_1)} : \cdots : e^{ik_N X^{n_N}(w_N)} :>_{j2} + < T_2(z) : e^{ik_1 X^{n_1}(w_1)} : \cdots : e^{ik_N X^{n_N}(w_N)} :>_{j1} = \\
= \ & \sum_{i=1}^{N} \left(\frac{\alpha k_i^2}{2} \frac{1}{(z-w_i)^2} + \frac{1}{z-w_i} \partial_{w_i} \right) <: e^{ik_1 X^{n_1}(w_1)} : \cdots : e^{ik_N X^{n_N}(w_N)} :> . \qquad (30)
\end{aligned}
$$

In (30) the index j1 (jump1) means that the corresponding stress-energy tensor is treated as if it commutes with the X^1 field and similarly for j2.

As a conclusion, we have shown that it is possible to deform nontrivially the closed bosonic string in two flat dimensions by following a quantization procedure based on a noncommutative harmonic oscillator. This deformation is shown to preserve sl(2,C) invariance of the four point function. Quantization using a more general noncommutative harmonic oscillator is currently under investigation.

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